Bernoulli's Scholium

In the scholium section of Bernoulli's proof he gives a correct argument that $\frac{W_{NR}}{W_{NR+N}}$ (Bernoulli's M/L) and $\frac{W_{NR}}{W_{NR-N}}$ (Bernoulli's M/Λ) can be made as large as desired by making N sufficiently large.

$$\frac{M}{L} = \frac{NRS+NS}{NRS-NR+R} * \frac{NRS+NS-S}{NRS-NR+2R} * \ldots * \frac{NRS+NS-(K-1)S}{NRS-NR+KR} * \ldots * \frac{NRS+S}{NRS}$$

Notice that the numerators are getting smaller and the denominators are getting larger as we go from left to right and that the rightmost fraction $\frac{NRS+S}{NRS} > 1$.

Therefore for K<N

$$\frac{NRS+NS}{NRS-NR+R} * \frac{NRS+NS-S}{NRS-NR+2R} * \ldots * \frac{NRS+NS-(K-1)S}{NRS-NR+KR} < \text{M/L}$$

and since the Kth term is smaller than the preceding terms

$$\left[\frac{NRS+NS-(K-1)S}{NRS-NR+KR}\right]^K < \text{M/L}$$

If K is such that $\left(\frac{R+1}{R}\right)^{\text{K}} \geq$ C(S-1), then if we can make $\frac{NRS+NS-(K-1)S}{NRS-NR+KR} \geq \frac{R+1}{R}$ Then M/L $\geq$ C(S-1).

Solving for N in the equation $\frac{NRS+NS-(K-1)S}{NRS-NR+KR} = \frac{R+1}{R}$ we get N $= \frac{K(T+1)-S}{R+1}$.

In Bernoulli's example R=30, S=20, C =1000 . So since T=R+S, T=50. C(S-1) = 19,000 and K = 301, since $\left(\frac{30+1}{30}\right)^{301}$ > 19,000.

So $N = \frac{301(50+1)-20}{30+1}$ > 494. So NT = 495∗50 = 24,750.

$$\frac{M}{\Lambda} = \frac{NRS+NR}{NRS-NS+S} * \frac{NRS+NR-R}{NRS-NS+2S} * \ldots * \frac{NRS+NR-(K-1)R}{NRS-NS+KS} * \ldots * \frac{NRS+R}{NRS}$$

So by the same reasoning as above we get

$$\left[\frac{NRS+NR-(K-1)R}{NRS-NS+KS}\right]^{K} < M/\Lambda$$

If K is such that $\left(\frac{S+1}{S}\right)^{K} \geq$ C(R-1) then if we can make $\frac{NRS+NR-(K-1)R}{NRS-NS+KS} \geq \frac{S+1}{S}$ Then $M/\Lambda \geq$ C(R-1) .

Solving for N in the equation $\frac{NRS+NR-(K-1)R}{NRS-NS+KS} = \frac{S+1}{S}$ we get

$N = \frac{K(T+1)-R}{S+1}$ So in Bernoulli's example, K is 211 since

$\left(\frac{20+1}{20}\right)^{211}$ > 29,000. So N = (211(50+1)-30)/(20+1) = 511.

So NT = 511x50 = 25,550. Since 25,550 > 24,750 if NT = 25,550 we're guaranteed that it will be at least 1000 times more likely that the relative frequency of successes lies in the range 3/5-1/50 through 3/5 + 1/50 than outside that range.

Comment

In my paper Bernoulli's Theorem Part One Lemma 8 it is shown that if M/L >= C(S-1) then the ratio of the sum of the number of ways of getting from NR + 1 successes to NR + N successes to the sum of the number of ways of getting more than NR + N successes will be greater than C. Similarly, in Part Two it is shown that if M/Λ >= C(R-1) then the ratio of the sum of the number of ways of getting NR - 1 through NR - N successes to the sum of the number of ways of getting less than NR - N successes will be greater than C.
So if N is such that M/L >= C(S-1) and M/Λ >= C(R-1), it is guaranteed that it will be at least C times more likely that the relative frequency of successes in NT trials will be in the range R/T - 1/T through R/T + 1/T than outside that range.

The reason Bernoulli uses $\frac{R+1}{R}$ for the equation $\frac{NRS+NS-(K-1)S}{NRS-NR+KR}=\frac{R+1}{R}$ is because he knew that this equation would have a solution for N.

Dividing the numerator and denominator on the left side of the equation by N gives $\frac{RS+S-(K-1)S/N}{RS-R+KR/N}$ The negative term in the numerator (K-1)S/N will approach zero as N increases and the positive term in the denominator KR/N will also approach zero, so Bernoulli knew the fraction would be increasing as N increases and would approach $\frac{RS+S}{RS-R}$ which is greater than $\frac{R+1}{R}$.

Similarly,the same reasoning shows that $\frac{NRS+NR-(K-1)R}{NRS-NS+KS}=\frac{S+1}{S}$ has a solution for N.